**MATH2050C Selected Solution to Assignment 10**

**Section 5.1** no. 3, 4ac, 5, 8, 10, 13.

(4a) The function $f(x) = [x]$ is continuous except at all integers.

(4b) The function $h(x) = [\sin x]$ is continuous whenever $\sin x$ is not equal to $-1, 0, 1$. At $x = 0$, $[\sin x] = 0$ for small $x > 0$ but $[\sin x] = -1$ for small $x < 0$, so it is not continuous at 0. Similarly, it is not continuous at all $n\pi$. On the other hand, $\sin x = 1$ if and only if $x = (2n + 1/2)\pi, n \in \mathbb{Z}$. For $x$ close to $(2n + 1/2)\pi$ from its right or left, $\sin x$ is very close to 1 but less than 1, so $[\sin x] = 0$, $h$ is not continuous at $(2n+1/2)\pi$. Similarly, it is not continuous at all $(2n + 1 + 1/2)\pi$. Conclusion: The discontinuity set of $h$ is $\{n\pi, (n + 1/2)\pi\}, n \in \mathbb{Z}$.

(5) We have

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}(x + 3) = 5 \ .$$

Therefore, the function $F(x) = f(x)$ when $x \neq 2$ and $F(2) = 5$ is a continuous function which extends $f$.

(13) Let $x_0$ be a continuity point of $g$. Let $\{x_n\}$ be a sequence of rational numbers tending to $x_0$. By continuity at $x_0$, $g(x_0) = \lim_{n\to\infty} g(x_n) = \lim_{x\to x_0} 2x_n = 2x_0$. On the other hand, let $\{y_n\}$ be an irrational sequence tending to $x_0$. We have $g(x_0) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty}(y_n+3) = x_0+3$. We get $2x_0 = x_0 + 3$ which implies $x_0 = 3$. Conclusion: 3 is the unique continuity point for $g$.

**Section 5.2** no. 1bc, 3, 7, 10, 11, 15.

(1b) $g$ is continuous on $[0, \infty)$. For, both $x$ and $\sqrt{x}$ are continuous functions on $[0, \infty)$, so is their sum $x + \sqrt{x} \in [0, \infty)$. As the function $y \mapsto \sqrt{y}$ is continuous on $[0, \infty)$, the composite function $g(x) = \sqrt{x + \sqrt{x}}$ is continuous on $[0, \infty)$.

(1c) $\sin x$ and the absolute value (function) are continuous on $(-\infty, \infty)$, so is their composite $|\sin x|$. It follows that $\sqrt{1 + |\sin x|}$ (the composite of $1 + |\sin x|$ and the square root function) is continuous on $(-\infty, \infty)$. As the quotient of two continuous functions is continuous away from where the denominator vanishes, we conclude that $h$ is continuous on $(-\infty, 0) \cup (0, \infty)$.

(7) Just let $f(x) = 1$ at rational $x$ and $f(x) = -1$ at irrational $x$.

(10) Let $P = \{x \in \mathbb{R} : \ f(x) > 0\}$ where $f$ is continuous everywhere. Let $c \in P$. By continuity, for $\varepsilon > 0$, there is some $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x, |x - c| < \delta$. Now, we choose $\varepsilon = f(c)/2$ and denote the corresponding $\delta$ by $\delta_0$. Then $|f(x) - f(c)| < f(c)/2$ implies $f(x) > f(c) - f(c)/2 = f(c)/2 > 0$ for all $x, |x - c| < \delta_0$. In other words, $(c - \delta_0, c + \delta_0) \subset P$. ( I do not use the notation $V_\delta(c)$.)

Note. This will be a commonly used fact.

(15) The formula

$$h(x) = \sup\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$$

is easily proved by considering each case $f(x) \geq g(x)$ or $f(x) \leq g(x)$. This formula clearly shows that $h$ is continuous whenever $f$ and $g$ are continuous.

Note. Can you find a corresponding formula for $\inf\{f, g\}$?